An operator approach to high-contrast homogenization

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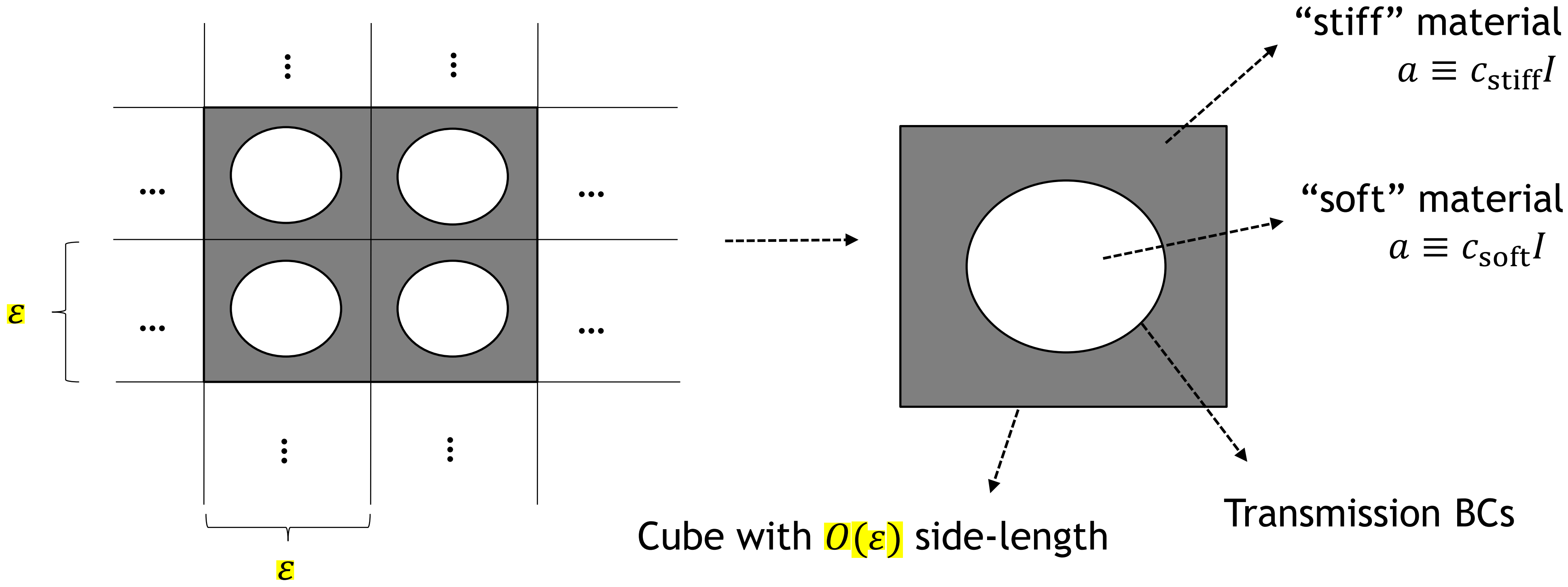
Problem Setup

$-\mathrm{div}(a(\cdot/\varepsilon)\nabla\cdot)$ "$\longrightarrow$" $-\mathrm{div}(a_{\mathrm{hom}}\nabla\cdot)$?

- Fix dimension $d \geq 2$. Consider the problem

$$-\mathrm{div}\left(\mathrm{a}\left(\frac{x}{\varepsilon}\right)\nabla u^{\varepsilon}\right) - \mu u^{\varepsilon} = f, \qquad f \in L^2(\mathbb{R}^d), \qquad \mu \in \mathbb{C}$$

- $a(x)$ is $\mathbb{Z}^d$-periodic, and looks like this:



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full space

- Operator POV
- Resolvent eqn $A_\varepsilon u^\varepsilon - \mu u^\varepsilon = \mathrm{f}$
- $\sigma(A_\varepsilon) \to \sigma(A_{\mathrm{hom}})$

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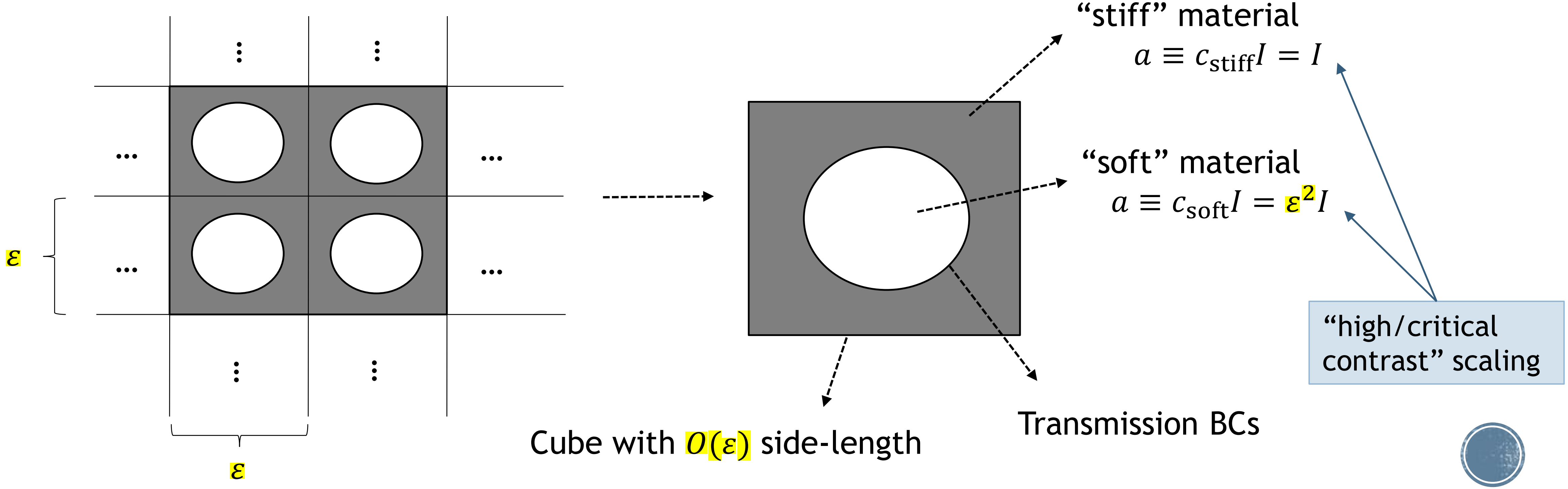


Methods available (non-HC)

(Murat 1978, Tartar 1979) Method of compensated compactness

$$U^\varepsilon \rightharpoonup U^0, V^\varepsilon \rightharpoonup V^0 \text{ in } \left(\mathrm{L}^2(\Omega)^{\mathrm{d}}\right)$$

$$\operatorname{div} U^\varepsilon \to f^0 \in H^{-1} \text{ and } \operatorname{curl} \mathrm{V}^\varepsilon = \boldsymbol{o}$$

Then $U^\varepsilon \cdot V^\varepsilon \rightharpoonup U^0 \cdot V^0$.

(Allaire 1992) Two-scale convergence method: We say $v^\varepsilon \xrightarrow{2} v^0$ if

$$\int_\Omega v^\varepsilon(x)\psi\left(x, \tfrac{x}{\varepsilon}\right)dx \to \iint_{\Omega\times[0,1]} v^0(x,y)\psi(x,y)\,dy\,dx$$

For all $\psi(x,y) \in \mathcal{D}(\Omega; C^\infty_{per}([0,1]))$.

Tartar's method of oscillating test functions (1977)

Γ-convergence, G-convergence, H-convergence, …

Two-scale expansion method

$$u^\varepsilon(x) = u_0\left(x, \tfrac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \tfrac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \tfrac{x}{\varepsilon}\right) + \cdots$$

(Birman-Suslina 2004) "spectral germ"

- Gelfand transform

$$A \cong \int_{[0,1]^d}^{\oplus} A(\tau)d\tau$$

- Perturbation theory

$$A(t)\varphi_n(t) = \lambda_n(t)\varphi_n(t), \qquad \tau = t\theta$$

- **Norm-resolvent** approximations!!!

Why norm-resolvent convergence?

- Let A_n and A be (unbounded) self-adjoint ops on a Hilbert space $\mathcal{H}$.
- We say that A_n converges to A in the norm-resolvent sense, denoted $A_n \overset{\text{nr}}{\to} A$, if

$$\|(A_n - \lambda)^{-1} - (A - \lambda)^{-1}\|_{op} \to 0, \qquad \text{as } n \to \infty, \qquad \text{for some } \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

- Implies strong convergence of solutions $u^n = A_n^{-1} f \to A_0^{-1} f = u^0$.
- (By functional calculus) $\|g(A_n) - g(A)\|_{op} \to 0,\ g \in C_0(\mathbb{R};\mathbb{C})$
- $A_n \overset{\text{nr}}{\to} A$ implies convergence of spectrum (in some sense), i.e.

$$\sigma\left(\text{nr} - \lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \sigma(A_n)$$

$$\sigma\left(\text{sr} - \lim_{n\to\infty} A_n\right) \subseteq \lim_{n\to\infty} \sigma(A_n)$$

$g(\lambda) = e^{it\lambda}$ not ok

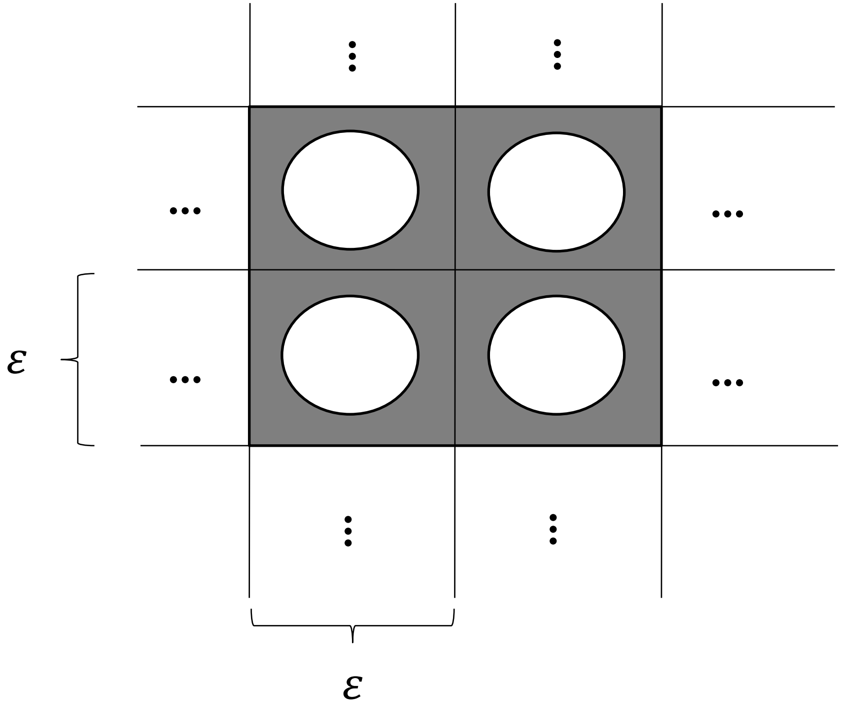
What it *cannot* achieve

- Spectral decomposition
- Might not have limits in general … norm resolvent asymptotics

Step 1

- From $\mathrm{A}_\varepsilon u^\varepsilon - \mu u^\varepsilon = f$, we apply a sequence of unitary transforms:

$$A_\varepsilon = G_\varepsilon^* \left(\int_{\varepsilon^{-1}Q'}^{\oplus} \Phi_\varepsilon^* A_\varepsilon^{(\varepsilon\theta)} \Phi_\varepsilon \, d\theta \right) G_\varepsilon$$



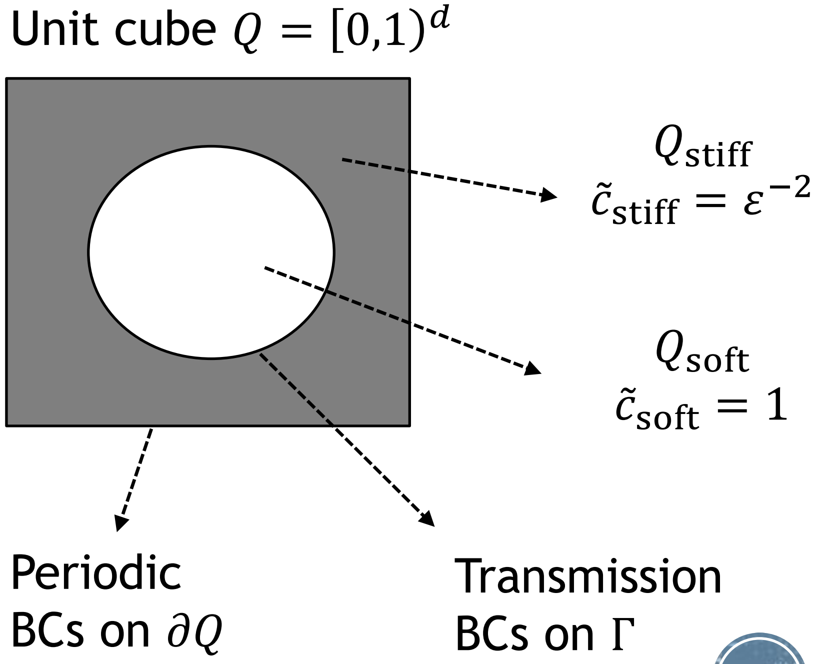
- Gelfand Transform

$G_\varepsilon : L^2(\mathbb{R}^d) \to L^2(\varepsilon^{-1}Q \times \varepsilon Q)$

(gives us a family of PDEs on $L^2(\varepsilon Q)$)

- Unitary rescaling

$\Phi_\varepsilon : L^2(\varepsilon Q) \to L^2(Q)$



Step 1

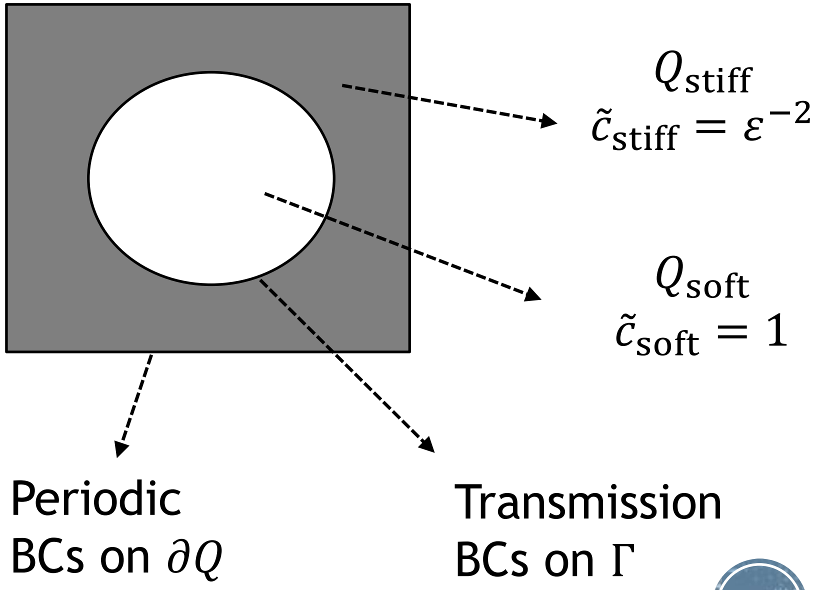
- Write $\tau = \varepsilon\theta \in Q' = [-\pi, \pi)^d$. The resolvent equation

$$\left(A_\varepsilon^{(\tau)} - z\right) u = f \in L^2(Q)$$

has a unique solution $u \equiv u_\varepsilon^{(\tau)} = u_{\text{soft}} + u_{\text{stiff}}$ whenever the following BVP can be solved uniquely in the weak sense:

$$\begin{cases} \varepsilon^{-2}\left(\frac{1}{i}\nabla + \tau\right)^2 u_{\text{stiff}} - z u_{\text{stiff}} = f, & \text{in } Q_{\text{stiff}}, \\ \left(\frac{1}{i}\nabla + \tau\right)^2 u_{\text{soft}} - z u_{\text{soft}} = f, & \text{in } Q_{\text{soft}}, \\ u_{\text{stiff}} = u_{\text{soft}}, & \text{on } \Gamma, \\ \varepsilon^{-2}\left[\frac{\partial u_{\text{stiff}}}{\partial n} + i(\tau\cdot n)u_{\text{stiff}}\right] + \left[\frac{\partial u_{\text{soft}}}{\partial n} + i(\tau\cdot n)u_{\text{soft}}\right] = 0, & \text{on } \Gamma, \\ u_{\text{stiff}} \text{ periodic} & \text{on } \partial Q \end{cases}$$

Unit cube $Q = [0,1)^d$

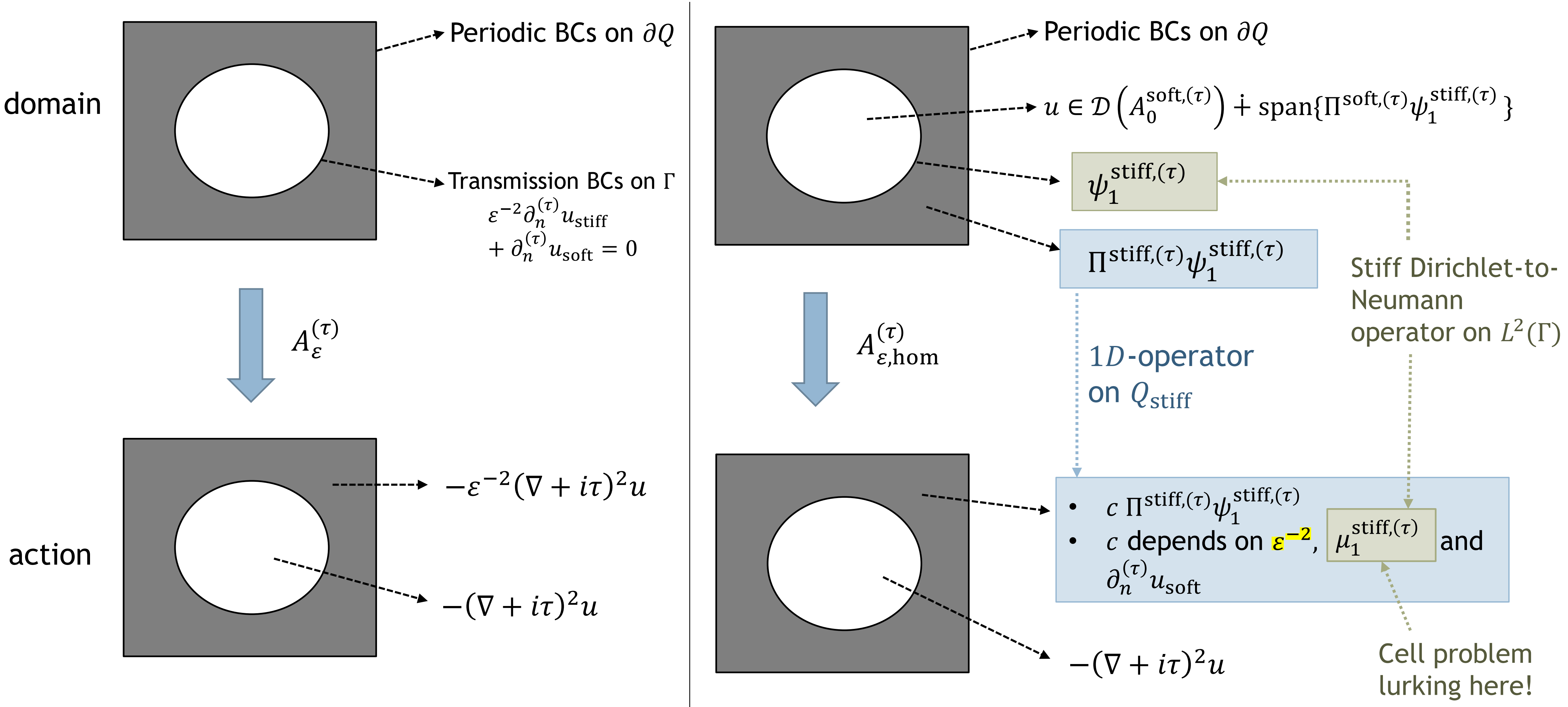


Our goal

- Find an operator $A_{\varepsilon,hom}^{(\tau)}$ that is
  - self-adjoint on a possibly smaller subspace $L^2(Q_{\text{soft}}) \oplus \widetilde{\mathcal{H}}$ of $L^2(Q)$.
  - Dependence on ε only allowed in the action of $A_{\varepsilon,hom}^{(\tau)}$ on the stiff component.

    (e.g. domain $\mathcal{D}\left(A_{\text{hom}}^{(\tau)}\right)$ cannot depend on ε.)
  - Is $O(\varepsilon^2)$ close to $A_{\varepsilon}^{(\tau)}$ in the norm-resolvent sense. $O(\varepsilon^2)$-error does not depend on τ.
- $A_{\varepsilon,hom}^{(\tau)}$ need not be unique since we are discussing asymptotics.

Result (as a picture)



Result

Theorem (Cherednichenko, Ershova, Kiselev 2020)

The operator $A^{(\tau)}_{\varepsilon,\mathrm{hom}}$ defined by

$$\mathcal{D}\left(A^{(\tau)}_{\varepsilon,\mathrm{hom}}\right) \coloneqq \{(u,\hat{u}) \in L^2(Q_{\mathrm{soft}}) \oplus \mathrm{span}\left\{\Pi^{\mathrm{stiff},(\tau)}\psi_1^{\mathrm{stiff},(\tau)}\right\}:$$

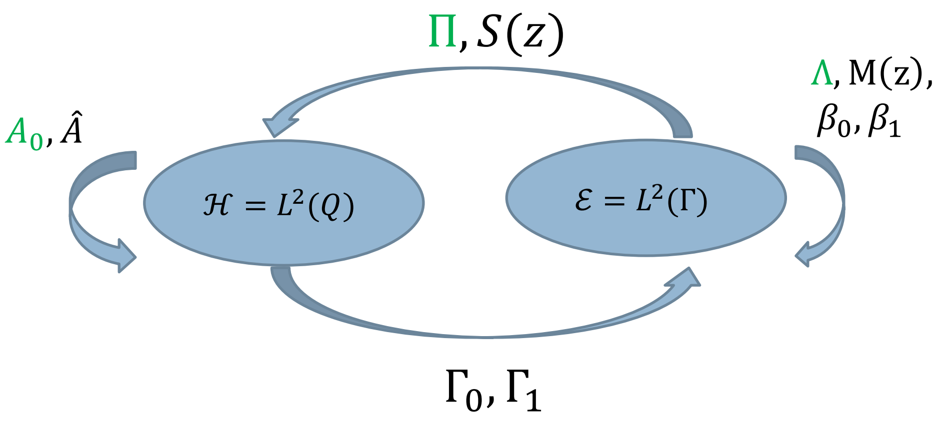
$$u \in \mathcal{D}\left(A_0^{\mathrm{soft},(\tau)}\right) \dot{+} \mathrm{span}\left\{\Pi^{\mathrm{soft},(\tau)}\psi_1^{\mathrm{stiff},(\tau)}\right\}, \quad \hat{u} = \Pi^{\mathrm{stiff},(\tau)}\Gamma_0^{\mathrm{soft},(\tau)}u\}$$

$$A^{(\tau)}_{\varepsilon,\mathrm{hom}}\binom{u}{\hat{u}} = \begin{pmatrix} -(\nabla + i\tau)^2 u \\ -\left(\breve{\Pi}^{\mathrm{stiff},(\tau)*}\right)^{-1}\mathcal{P}^{(\tau)}\left[\partial_n^{(\tau)}u|_\Gamma + \varepsilon^{-2}\mu_1^{\mathrm{stiff},(\tau)}u|_\Gamma\right]\end{pmatrix}$$

is self-adjoint on $L^2(Q_{\mathrm{soft}}) \oplus \mathrm{span}\left\{\Pi^{\mathrm{stiff},(\tau)}\psi_1^{\mathrm{stiff},(\tau)}\right\}$, and is $O(\varepsilon^2)$ close to $A_\varepsilon^{(\tau)}$ in the norm-resolvent sense.

This estimate is uniform in $\tau \in Q'$ and $z \in K_\sigma$ (a compact set $\sigma > 0$ distance away from the real line.)

Boundary triples



A (Ryzhov) boundary triple (A_0, Λ, Π) needs:

- Separable Hilbert spaces $\mathcal{H}$ and $\mathcal{E}$.
- (Dirichlet operator) A_0 an unbounded SA op on $\mathcal{H}$, with $0 \in \rho(A_0)$.
- (DTN operator) Λ an unbounded SA op on $\mathcal{E}$.
- (Lift) $\Pi: \mathcal{E} \to \mathcal{H}$, a bounded injective linear map.
- $\mathcal{D}(A_0) \cap \operatorname{ran}(\Pi) = \{0\}$

This gives us meaning to $(\hat{A}_{\beta_0,\beta_1} - z)u = f \in \mathcal{H}$, or equivalently

$$\begin{cases} (\hat{A} - z)u = f \\ (\beta_0\Gamma_0 + \beta_1\Gamma_1)u = 0 \end{cases}$$

with a nice formula on the resolvents:

$$R_{\beta_0,\beta_1}(z) = (A_0 - z)^{-1} - S(z)\left(\overline{\beta_0 + \beta_1 M(z)}\right)^{-1}\beta_1 S(\bar{z})^*$$

Approx $\Lambda = \sum \mu_k \langle \cdot, \psi_k \rangle \psi_k$

by $\mu_1 \langle \cdot, \psi_1 \rangle \psi_1$

Thank you!